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# THE $(\mathbf{1},\overline{\mathbf{2}})$-INTERSECTION INDEX OF A GRAPH WITH LARGE MINIMUM DEGREE AND ITS APPLICATION IN CRISIS MANAGEMENT

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**Purpose:** The purpose of this paper is to prove the conjecture which states that the $(1,\overline{2})$-intersection index of a graph with $\delta(G) \geq 3$ is equal to zero. Moreover, practical applications of this result are given.
**Design/methodology/approach**: We prove the conjecture by considering cases and indicating in each case two disjoint sets such that one of them is a $(1,1)$-dominating set and the second one is a proper $(1,2)$-dominating set. We also use a graph to model the problem of storage of supplies in Poland in case of a crisis.
**Findings:** For every connected graph, in which every vertex has at least three neighbors, the $(1,\overline{2})$-intersection index is equal to zero. It ensures that in most cases there exists an optimal allocation of water and food supplies throughout a given region.
**Practical implications:** The findings may be used by crisis management services when planning the allocation of food and water reserves.
**Originality/value:** The obtained results and applications are new and original, they may be of value to mathematician working in the field of theoretical and applied graph theory, as well as to researchers working on crisis management and methods of efficient supply allocations.
**Keywords:** graph, $(1,2)$-dominating set, $(1,\overline{2})$-intersection index, storage of supplies in crisis.
**Category of the paper:** Research paper.

## 1. Introduction

In (Hedetniemi et al., 2008) the authors introduced and studied the concept of secondary dominating sets in graphs. Let $k$ be a positive integer. The subset $D$ of vertices in a graph $G$ is $(1,k)$-*dominating* if for every vertex $x \in V(G) \setminus D$ there exist two vertices $y, z \in D$ such that $xy \in E(G)$ and $d_G(x,z) \leq k$. For $k = 1$ these sets are equivalent to the double dominating sets, which were extensively studied since 1985, see for example (Bednarz, Pirga, 2024; Cabrera-

Martinez, Estrada-Moreno, 2023; Fink, Jacobson, 1985; Harant 2005). The case when $k = 2$ is also studied in the literature, see (Hedetniemi et al., 2008; Michalski, Bednarz, 2021; Michalski, Włoch, 2020; Raczek, 2024). In 2022 Michalski et al. defined *proper* $(1,2)$*-dominating sets* as sets which are $(1,2)$-dominating but not $(1,1)$-dominating. They studied the problem of the existence and the minimum cardinality of these sets in graphs. Proper $(1,2)$-dominating sets are also denoted as $(1,\overline{2})$-dominating sets.

One of obtained results (Michalski et al., 2022) was the complete characterization of connected graphs having a proper $(1,2)$-dominating set.

**Theorem 1.1** (Michalski et al., 2022). *A connected graph $G$ has a proper $(1,2)$-dominating set if and only if $G$ is not a complete graph.*

From Theorem 1.1 we can immediately derive the following corollary concerning disconnected graphs.

**Corollary 1.2.** *A disconnected graph $G$ has a proper $(1,2)$-dominating set if and only if at least one of components of $G$ is not a clique.*

As a continuation of research concerning $(1,\overline{2})$-dominating sets, in (Kosiorowska et al., 2023) the authors introduced a special graph parameter, called a $(1,\overline{2})$-intersection index, which tells us "how much disjoint" can be a $(1,\overline{2})$-dominating set and a $(1,1)$-dominating set in a graph.

Formally, let $\mathcal{F}_{(1,1)}$ be a family of all $(1,1)$-dominating sets of a graph $G$ and let $\mathcal{F}_{(1,\overline{2})}$ be a family of all proper $(1,2)$-dominating sets of $G$. Then let us denote

$$\sigma(G) = \min_{D \in \mathcal{F}_{(1,1)}, D^* \in \mathcal{F}_{(1,\overline{2})}} |D \cap D^*|.$$

The number $\sigma(G)$ is called a $(1,\overline{2})$-*intersection index* of a graph $G$. Of course this parameter is defined only for graphs which have a $(1,\overline{2})$-dominating set - their characterizations were given in Theorem 1.1 and Corollary 1.2.

In (Kosiorowska et al., 2023) the authors determined the value of $\sigma(G)$ in some classes of graphs and gave a sufficient condition for a tree $T$ to satisfy the equality $\sigma(T) = 0$. At the end of the paper some conjectures were given, including the following.

**Conjecture 1.3** (Kosiorowska et al., 2023). Is $\sigma(G) = 0$ providing $\delta(G) > 2$ and $|V(G)|$ is large?

If this conjecture was true, it would mean that in a graph $G$ such that $\delta(G) > 2$ we can always find two disjoint subsets of $V(G)$ such that one of them is a proper $(1,2)$-dominating set and the second one is $(1,1)$-dominating set. In this paper we will prove Conjecture 1.3 in the form given in Theorem 2.1. For now, due to simplicity reasons, we restrict ourselves to the case of connected graphs.

## 2. Main results

**Theorem 2.1.** Let $G \neq K_n$ be a connected graph. If $\delta(G) \geq 3$ then $\sigma(G) = 0$.

*Proof.* Let $S$ be a maximal independent set of $G$. We consider the following cases.

1) $|S| = 1$.

   Then $v \in S$ is adjacent to any other vertex. Otherwise, $S$ would not be maximal. Since $G$ is not a complete graph, there exist two vertices $a_1, a_2$ which are not adjacent to each other. Then $S^* = S \cup \{a_1\}$ is a $(1,2)$-dominating set and $a_2$ has exactly one neighbour in $S^*$. Hence $S^*$ is a $(1,\overline{2})$-dominating set. Since $\delta(G) \geq 3$, vertices $v$ and $a_1$ have at least two neighbours in the set $\overline{S} = V(G) \setminus S^*$. Thus $\overline{S}$ is $(1,1)$-dominating. The sets $S^* \cap \overline{S} = \emptyset$, so $\sigma(G) = 0$.

2) $|S| \geq 2$.

   Let us consider the following subcases.

   a) $S$ is a $(1,\overline{2})$-dominating set.

      Since $\delta(G) \geq 3$ and $S$ is independent, every vertex from the set $S$ has at least three neighbours from the set $S' = V(G) \setminus S$. Thus $S'$ is 3-dominating, so also $(1,1)$-dominating. The sets $S \cap S' = \emptyset$, so $\sigma(G) = 0$.

   b) $S$ is a $n$-dominating set, $n \geq 2$.

      Let $k = \min\{\, |N(v) \cap S| : v \in V(G) \setminus S\}$ and $a \in V(G) \setminus S$ be a vertex with exactly $k$ neighbours in $S$. Consider a set $S^* = \big(S \setminus (N(a) \cap S)\big) \cup \{a, x\}$, where $x \in N(a) \cap S$. We claim that $S^*$ is a $(1,\overline{2})$-dominating set. We can partition the set $V(G) \setminus S^*$ into two disjoint sets as $S_1, S_2$ such that $S_1$ consists of all vertices which belong neither to $S$ nor $S^*$ and $S_2$ consists of all vertices which belong to $S \setminus S^*$. Let $y \in S_1$. If $N(y) \cap N(a) \cap S = \emptyset$, then $y$ is still at least $k$-dominated by $S^*$. Otherwise, if $y$ and $a$ have $p$ common neighbours belonging to $S$, $1 \leq p \leq k$, then $y$ is dominated either by $x$ or by $y_1 \in \big(N(y) \setminus N(a)\big) \cap S$. Moreover, $y$ is 2-dominated or within the distance at most 2 from $a$ so $y$ is $(1,2)$-dominated by $S^*$. Let $u \in S_2$. Then $N(u) \cap S^* = \{a\}$ and by independence of the set $S$ we have $d_G(u,x) = 2$. Hence $S^*$ is a $(1,\overline{2})$-dominating set.

      Let $(S^*)' = V(G) \setminus S^*$. Every component of the induced subgraph $G[S^*]$ may have at most two vertices. Therefore, every vertex from $S^*$ has at most one neighbour in $S^*$. Since $\delta(G) \geq 3$ then every vertex from $S^*$ must have at least two vertices in $(S^*)'$. Hence $(S^*)'$ is a 2-dominating set. Of course $S^* \cap (S^*)' = \emptyset$, so $\sigma(G) = 0$.

c) $S$ is not a $(1,2)$-dominating set.

Since $S$ is a maximal independent set then $S$ is dominating. Therefore there are vertices in $V(G) \setminus S$ which have exactly one neighbour in $S$ and the distance from other vertices in $S$ is greater than 2. Let us denote the set of such vertices by $R$ and let $S \cap N(R) = \{x_1, x_2, \dots, x_q\}$. Each vertex from $R$ has exactly one neighbour in the set $S \cap N(R)$. Hence we can partition the set $R$ into disjoint non-empty sets $R_1, R_2, \dots, R_q$ where $R_i = N(x_i) \cap R$, $1 \leq i \leq q$. Let us consider another two subcases.

i) There exists $k$ such that $|R_k| \geq 2$.

Let choose from each set $R_i$ one vertex and denote it by $r_i$. Then consider the set $S^* = S \cup \bigcup_{i=1}^{q} \{r_i\}$. For every vertex $z$ from $R \setminus S^*$ there exists a path $z - x_l - r_l$, so all such vertices are $(1,2)$-dominated by $S^*$. The set $S^*$ is $(1,2)$-dominating. Moreover, all vertices from $R_k \setminus S^*$ have exactly one neighbour in $S^*$. Hence $S^*$ is a $(1,\overline{2})$-dominating set.

ii) $|R_i| = 1$ for all $i = 1,2,\dots,q$.

Since $\delta(G) \geq 3$, for all $i$ we have that the vertex $r_i$ has exactly one neighbour in $S$ - the vertex $x_i$, and at least two neighbours not in $S$, let us denote them as $t_i^1, t_i^2$, which are also adjacent to $x_i$, because otherwise $r_i$ would be $(1,2)$-dominated by $S$. Let us consider the set $S^* = S \cup \bigcup_{i=1}^{q} \{t_i^1\}$. This set is (1,2)-dominating. Moreover, the vertices $t_i^2$ have exactly one neighbour in $S^*$ for all $i = 1, 2, \dots, q$. Hence, the set $S^*$ is a $(1,\overline{2})$-dominating set.

In each of two above subcases (2c(i), 2c(ii)) let us define $(S^*)' = V(G) \setminus S^*$. Every component of the induced subgraph $G[S^*]$ may have at most two vertices, so every vertex from $S^*$ has at most one neighbour in $S^*$. Since $\delta(G) \geq 3$ then every vertex from $S^*$ must have at least two vertices in $(S^*)'$. Hence $(S^*)'$ is a $(1,1)$-dominating set. Of course $S^* \cap (S^*)' = \emptyset$, so $\sigma(G) = 0$.

In all cases we can find a $(1,\overline{2})$-dominating set and $(1,1)$-dominating set which are disjoint. Hence for every connected, non-complete graph $G$ we obtain $\sigma(G) = 0$.

## 3. Applications

Let us imagine that in case of emergency or crisis we want to build big warehouses in each voivodeship to store food and water in Poland. Moreover, each warehouse should store only one type of supplies (food or water) because otherwise a warehouse containing both types of supplies would be more vulnerable to a potential attack. We want to assure that the following conditions are satisfied:

- if there is no water warehouse in a given voivodeship than there should be such warehouses in at least two neighboring voivodeships,
- if there is no food warehouse in a given voivodeship than there should be such a warehouse in one neighboring voivodeship and the next food warehouse should not be further than two voivodeships away,
- in each voivodeship we want to build at most one warehouse to minimize the potential damage in case of any crisis in a given voivodeship.

The first two assumptions follow from the fact that we want to always have a backup warehouse and that water is more crucial for survival than food, so the warehouses with food might be more scarcely located.

It is clear that this problem can be modelled by a graph, where:

- we identify each voivodeship with a vertex,
- two vertices are adjacent if and only if the corresponding voivodeship are neighbors,
- the set of voivodeship with water warehouses corresponds to a $(1,1)$-dominating set,
- the set of voivodeship with food warehouses corresponds to a $(1,2)$-dominating set.

In Figure 1 we can see the administrative division of Poland into voivodeships (for clarity we use only abbreviations rather than full names). Moreover, there is also a graph model of this division created according to assumptions given above.

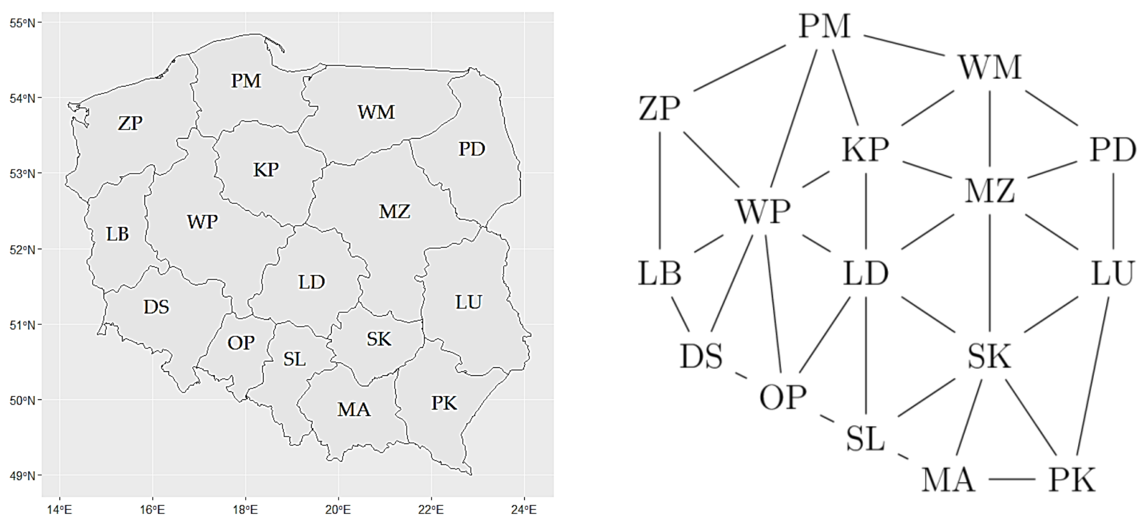


**Figure 1.** Division of Poland into voivodeships and its graph model.

Thanks to Theorem 2.1 we know that the $(1,\overline{2})$-intersection index of our voivodeship graph is equal to zero, because every voivodeship has at least three neighbours. Hence, we can find locations of food and water warehouses satisfying desired conditions. In Figure 2 we present the exemplary solution of this problem. The locations of water warehouses are denoted by blue ellipses, while food warehouses are denoted by red rectangles.

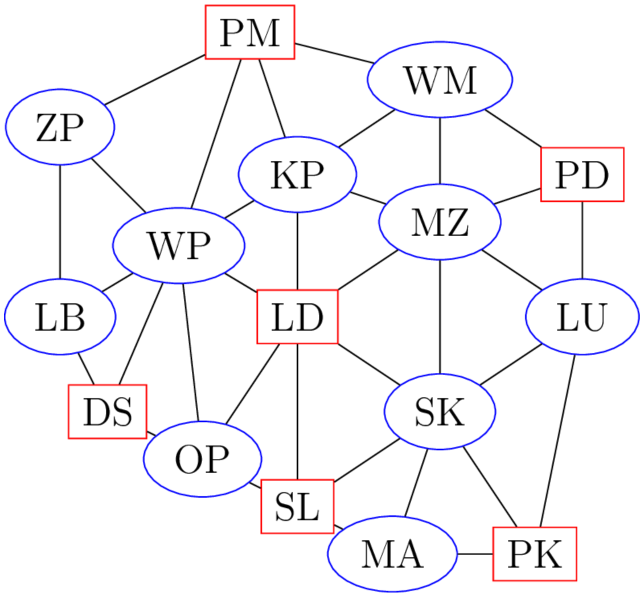


**Figure 2.** Exemplary location of water and food warehouses.

Let us notice that Theorem 2.1 can be applied not only to Poland but also to solve a similar problem in any part of the world. It is possible due to the fact that very rarely a region in the map has less than three neighboring regions.

## 4. Conclusions

Theorem 2.1 gives the exact value of $(1,\overline{2})$-intersection index in connected graphs with $\delta(G) \geq 3$. More precisely, for all such graphs we have $\sigma(G) = 0$. Additionally, by Corollary 1.2, we can immediately extend Theorem 2.1 to disconnected graphs.

**Corollary 4.1.** *Let $G$ be a disconnected graph such that $\delta(G) \geq 3$ and at least one of the components of $G$ is not a clique. Then $\sigma(G) = 0$.*

Summing up, in this paper we completely solved the problem of determining the value of $(1,\overline{2})$-intersection index in graphs with minimum degree greater than two. We also presented an application of this theoretical problem in the field of crisis management, basing on the example of storing water and food supplies in Poland.

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