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MONTE CARLO SIMULATION ANALYSIS OF THE PERT METHOD FOR COMPLETE GRAPH WITH ALL ACTIVITIES AS CRITICAL

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Purpose: The main objective of this study is to conduct a time analysis on a complete PERT network in a situation where all activities in the network are critical. This analysis is more exploratory and theoretical in nature, as it assumes a very specific case of a project and the potential implications arising from it.
Design/methodology/approach: The analysis was performed on the full PERT network including 6 events and the resulting number of 15 activities. The numerical procedure was carried out by: determining the number of events and parameters of the project duration, determining the (maximum) number of activities - determining the parameters of the distribution of activity durations using mathematical programming, determining the number of iterations, in each iteration: generating activity durations, determining critical paths, determining time duration of the project and analysis of the results obtained.
Findings: The work draws three main conclusions: the distribution of the project duration differs significantly from the theoretical PERT time, the theoretical activity durations affect the critical importance of activities in the project implementation, the number of events in the critical path affects the project implementation deadline.
Research limitations/implications: The obtained results depend on the adopted methodology, in particular the numerical procedure for generating times: optimistic, modal, pessimistic of activities and generating activity durations from a normal distribution. Further research will focus on these issues.
Originality/value: the main novelty of the work is the analysis using Monte Carlo simulation on the full PERT network, where all activities are critical.

Keywords: Monte Carlo simulation, PERT method, complete graph.

Category of the paper: Research paper.

1. Introduction

The PERT method (The Program Evaluation and Review Technique) (Malcolm et al., 1959; Cook, 1966; Trocki et al., 2003) is a network method of planning and controlling the implementation of the project. It is an extension of the CPM (Critical Path Method) method by

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adding uncertainty related to the implementation time of particular activities, and thus the implementation of the entire project (project duration). This method takes into account the risk associated with performing tasks by taking into account three types of time - the shortest possible (optimistic), the most probable (modal) and the potentially longest time (pessimistic). This method assumes that the implementation time of the entire project is the sum of the expected implementation times of critical activities. The expected duration of an activity is calculated from the formula:

$$te = \frac{a+4b+c}{6}, \tag{1}$$

where:
a is the optimistic time,
b is the most probable time,
c is the pessimistic time.

The project implementation time variance is determined as the sum of the variances of activities on the critical path. In turn, the activity duration variance is determined from the formula:

$$var = \frac{(c-a)^2}{36}. \tag{2}$$

Usually, when analyzing the implementation time of a project, it is assumed that the time distribution is consistent with the normal distribution as a special case of the beta distribution - PERT distribution.

In a situation where the project does not consist of activities that can be implemented in parallel, the distribution of project implementation time deviates from the normal distribution.

This is a consequence of the fact that the project implementation time is described by a random variable which is the maximum value of the sum of random variables of the times of individual activities creating the critical path. Analyzes based on classic PERT should therefore be treated as an approximation.

In a general approach, the solution to this problem may be the use of a simulation approach (Van Slyke, 1963; Lu AbouRizk, 2000; Wyrozębski, Wyrozębska, 2013; Walczak, 2014; Karabulut, 2017; Salas-Morera et al., 2018). Moreover, simulation analysis is advisable due to different approaches to estimating task completion times (Udoumoh, Ebong, 2017; Deshmukh, Rajhans, 2018). Therefore, a Monte Carlo simulation approach was used in this work.

In the PERT method, the logical structure of the project is presented by a network (graph) of dependencies. The nodes of the network are events (milestones), while the edges of the network symbolize activities in the project (tasks).

A complete graph (Fully Connected Network) is a network in which every vertex is connected to every other vertex. In terms of the PERT method, this means a project that has the maximum possible number of tasks that make up the entire project. Such a network was adopted in this work because it is a universal project model - any project can be modelled using such a network, because some edges in the graph may symbolize dummy activities.

As can be seen, the main problem (and deviations from the normal project implementation time classically postulated in the PERT method) lies in the potential occurrence of many critical paths (Dodin, 1984; Soroush, 1994). This situation occurs when the project consists of activities, at least one of which can be performed in parallel.

From this point of view, an interesting issue is the analysis of project implementation time when all activities in the network are critical. Such a theoretical project is characterized by “tight in time”. Any deviation from the expected completion time of any activity (task) has a potentially significant impact on the project completion date.

Additionally, the properties of the full network and the critical nature of each task may generate interesting implications with respect to task implementation and design.

For example, comparing the critical path including all milestones (events) and the path containing only the first and last event (beginning and end of the project) - in the sense of the PERT method, both paths (like all paths) have the same parameters (expected value and variance). Both of these paths are independent (they do not have common tasks). From the point of view of time analysis, they are identical. The difference is in the number of tasks performed. Comparing the implementation time of these paths leads to a comparison of the implementation time of a project consisting of one task and the implementation of a project when this task is divided into subtasks. Continuing, the research question arises: whether, in accordance with practical recommendations, it is important to divide large tasks into smaller ones.

Taking into account the above considerations, the main goal of the work is time analysis on the full PERT network in a situation where all activities in the network are critical. This analysis is more exploratory and theoretical in nature as it assumes a very specific case of design and the potential implications arising from it. The following research questions were formulated due to the nature of the experiment:

1. Does the distribution of implementation time of the tested project differ significantly from the theoretical PERT time?
2. Do theoretical activity durations influence the critical importance of activities in project implementation (in simulations)?
3. Does the number of events in the critical path affect the project completion date?

1. Methods

1.1. Model

In the analyzed example of the PERT network, 6 events and the resulting number of 15 activities were assumed. The logical structure of the network is shown in Figure 1.

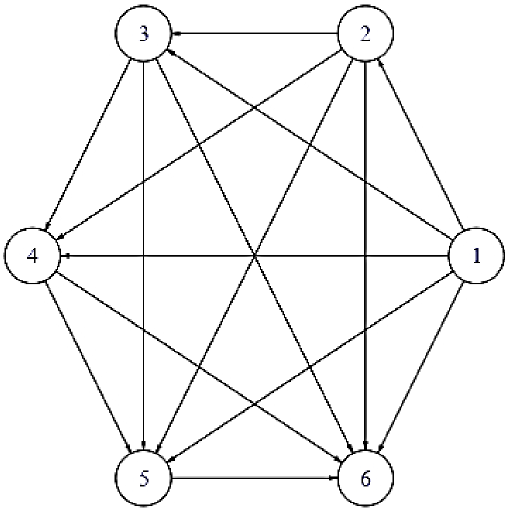

Figure 1. Logical structure of the analyzed network.

Table 1 shows the parameters of the activity time distribution between events (i, j). Where i is the number of the preceding event and j is the number of the event following the action. Expected time to complete the activity (te) and time variance (var). The activity duration parameters were set so that each path was a critical path. The expected duration of the project is 100 (time units), variance 10.

Table 1.
Expected time and variance of the activities duration of the analyzed network

i	j	optimistic	modal	pessimistic	te	var
1	2	13,743	21,052	22,052	20,000	1,918
1	3	32,322	41,336	42,336	40,000	2,785
1	4	50,138	61,772	62,772	60,000	4,434
1	5	68,971	82,006	83,006	80,000	5,471
1	6	84,855	102,829	103,829	100,000	10,000
2	3	16,010	20,598	21,598	20,000	0,867
2	4	32,735	41,253	42,253	40,000	2,516
2	5	51,241	61,551	62,551	60,000	3,553
2	6	66,452	82,509	83,509	80,000	8,082
3	4	14,246	20,951	21,951	20,000	1,649
3	5	32,472	41,305	42,305	40,000	2,686
3	6	47,237	62,353	63,353	60,000	7,215
4	5	15,575	20,685	21,685	20,000	1,037
4	6	28,871	42,026	43,026	40,000	5,566
5	6	10,026	21,795	22,795	20,000	4,529

1.2. Simulation

A single iteration consisted of generating activity durations, determining the duration of the project and critical paths. 100000 iterations were performed.

The obtained results were analyzed. Simulations and calculations were performed in the R environment (R Core Team, 2022).

The numerical procedure was carried out in the following generalized steps:

1. Determining the number of events (6 events assumed), the expected project duration (100 time units assumed) and the project duration variance (10 square time units).
2. Determining the (maximum) number of activities - connections between all events.
3. Determining the parameters of the distribution of activity durations (optimistic, modal, pessimistic) using mathematical programming.
4. Determining the number of iterations (n = 100000).
5. In each iteration: generating activity durations (using normal distribution), determining critical paths, determining the duration of the project.
6. Analysis of the obtained results.

2. Results

The general numerical characteristics of the project implementation time are presented in Table 2.

Table 2.
Basic statistical characteristics of the performed simulation

statistics	**values**
Mean	104,728
Std.Dev	2,105
Min	96,591
Q1	103,265
Median	104,644
Q3	106,080
Max	114,731
MAD	2,085
IQR	2,815
CV	0,020
Skewness	0,259
SE.Skewness	0,008
Kurtosis	0,142
N.Valid	100000

The analysis of the values presented in Table 2 indicates, first of all, that the main parameters of the project implementation time distribution differ from the theoretical PERT values. The expected implementation time is significantly longer, but the variability measured by variance is lower. The exact time distribution obtained in the experiment compared to the theoretical PERT is shown in Figure 2.

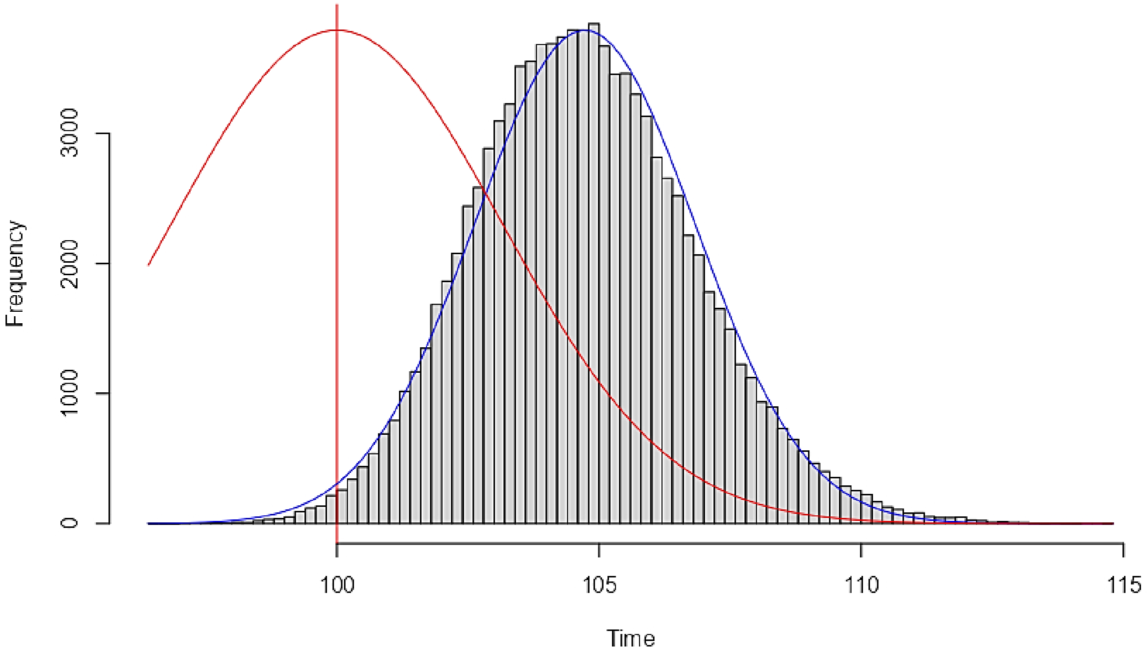

Figure 2. Distribution of project implementation time in the experiment.

The histogram presented in Figure 2 presents the distribution of project implementation time resulting from the conducted experiment. The blue line shows the theoretical distribution of the project implementation assuming normality of distribution and parameters obtained from the simulation N(104.728; 2.105). The red line shows the theoretical time distribution resulting from the PERT method - a normal distribution with an expected value of 100 and variance of 10.

The estimated probability of meeting the PERT expected time (100 time units) is very small and is approximately 0.007. However, the distribution of project implementation time is not consistent with the normal distribution (Jarque-Bera Normality Test, JB = 1205, p-value < 0.001).

The results regarding the frequency of occurrence of individual activities in the simulations are presented in Table 3.

Table 3.
Frequencies of critical activities in the PERT network and the simulations performed

i	j	how many times critical	how many times critical in simulations	te	var
1	2	8	43515	20	1,918
1	3	4	23395	40	2,785
1	4	2	13466	60	4,434
1	5	1	7916	80	5,471
1	6	1	11743	100	10
2	3	4	18308	20	0,867
2	4	2	9605	40	2,516
2	5	1	5412	60	3,553
2	6	1	10190	80	8,082

Cont. table 3.

3	4	4	17415	20	1,649
3	5	2	8969	40	2,686
3	6	2	15319	60	7,215
4	5	4	16703	20	1,037
4	6	4	23783	40	5,566
5	6	8	39000	20	4,529

The value "how many times critical" refers to the share of a given activity in the PERT network and "how many times critical in simulations" refers to the occurrence of a given activity as critical in simulations.

The analysis of the data contained in Table 1 and Table 3 indicates the existence of a relationship between the expected time of performing an activity (te) and the values: "how many times critical" (corr = -0.768, t = -4.324, p < 0.001) and "how many times critical in simulations" (corr = -0.591, t = -2.642, p = 0.020). In this situation, the conclusion (alpha = 0.05) that the longer the expected time to complete an activity, the less likely it is that the activity is critical, is justified. This is due to the structure of the activity network and the significant correlation between the expected duration of the activity (te) and the variable "how many times critical" (corr = -0.768, t = -4.324, p < 0.001).

The interdependence between "how many times critical" and "how many times critical in simulations" (corr = 0.961, t = 12574, p < 0.001) can be considered as a certain disturbance in these relations. Moreover, the PERT network was constructed in such a way that all activities were critical (in terms of expected activity execution times). However, the results of the simulations indicate that it is unlikely that there will be more than one critical path in a specific project implementation. The results regarding the occurrence of specific critical paths are presented in Table 4.

Table 4.
Data and results regarding the critical paths that occur

Path	how many times critical in the simulation	expected time	time variance	number of events
c(1, 2, 3, 4, 5, 6)	2903	100	10	6
c(1, 2, 3, 4, 6)	4460	100	10	5
c(1, 2, 3, 5, 6)	3848	100	10	5
c(1, 2, 3, 6)	7097	100	10	4
c(1, 2, 4, 5, 6)	3892	100	10	5
c(1, 2, 4, 6)	5713	100	10	4
c(1, 2, 5, 6)	5412	100	10	4
c(1, 2, 6)	10190	100	10	3
c(1, 3, 4, 5, 6)	4102	100	10	5
c(1, 3, 4, 6)	5950	100	10	4
c(1, 3, 5, 6)	5121	100	10	4
c(1, 3, 6)	8222	100	10	3
c(1, 4, 5, 6)	5806	100	10	4
c(1, 4, 6)	7660	100	10	3
c(1, 5, 6)	7916	100	10	3
c(1, 6)	11743	100	10	2
SUM	100035			

Out of 100,000 simulations, a maximum of 35 cases had more than one critical path. Therefore, it is unlikely that more than one critical path will occur in the implementation of the project. However, this is strongly dependent on the rounding adopted - the accuracy of time measurement.

Whether a given path is critical depends on the number of events (corr = -0.943, t = -10.569, $p < 0.001$). Generally, the greater the number of events in a path (activities/tasks), the lower the risk that a given path will be critical.

3. Discussion and conclusions

The first question asked: "Does the distribution of implementation time of the studied project differ significantly from the theoretical PERT time?" can be answered positively. Distribution analysis also shows why projects are "always" late (Schonberger, 1981). As the experiment shows, this is especially visible in the case of "tight" projects in which all activities are critical. The expected project completion time (104.728) is significantly different from the theoretical expected PERT time (p-vaue < 0.001). The distribution of project duration determined in the course of the experiment also differs significantly from the normal distribution. Therefore, it can be recommended that in the case of analyzing projects with numerous critical paths, the use of the normal distribution is not justified. The risk of failing to meet the directive deadline will be much higher than that resulting from a normal distribution (for the theoretical parameters of PERT).

The answer to the second question: "Do theoretical activity durations influence the critical importance of activities in project implementation (in simulations)?" is not clear. The experimental results indicate that the answer should be positive, but the structure of the activity network implies the existence of such a relationship. It should be emphasized that the theoretical durations of activities were generated during the experiment in such a way that all paths and activities were critical. It can be seen that there is a strict linear relationship between the modal time of an activity (modal) and the pessimistic time (Pearson's correlation coefficient is 1). These times, in turn, are used to determine the values "te" and "var" from formulas (1) and (2). Taking this into account, the answer to the second question is positive, as shown by the results, but this is a direct result of the assumptions made in the construction of the experiment. Taking into account the results obtained, a recommendation can be made that in the case of "tight" projects, there will always be such a relationship and more attention should be paid to activities whose expected duration is shorter.

The answer to the third question "Does the number of events in the critical path affect the project completion date?" is also positive. In this case, the greater the number of activities, the lower the risk that the path will be critical. A larger number of activities indicates a shorter expected completion time on average because the expected PERT time for all paths is identical. These results are in some opposition to the conclusions resulting from the answer to the second question. This is a consequence of the fact that activities with shorter implementation times enter a larger number of critical paths. Comparing the path consisting of one activity 1-6 and the path with the largest number of activities 1-2-3-4-5-6, one can notice a large disproportion in the estimated risk. Path 1-6 was critical in 11.7% of cases, while path 1-2-3-4-5-6 was critical in only 2.9% of cases. Since both of these paths are independent, it can be directly observed that the number of activities has an impact on the project implementation time. Referring to the example from the introduction, we can also indirectly conclude that it is justified to divide "large" tasks into smaller subtasks.

To sum up, it can be concluded that in terms of the experiment conducted and the type of activity dependency network examined:

1. the implementation time distribution of the studied project differs significantly from the theoretical PERT time.
2. theoretical activity durations influence the critical importance of activities in project implementation (in simulations).
3. the number of events in the critical path affects the project completion date.

It is worth noting that the proposed Monte Carlo analysis on the full network can be adapted to any part of the project, because some connections (edges) between nodes may be dummy activities. Relatively, it can be adapted to parts of larger projects, in that part of the tasks that are interdependent and "tight" in nature.

Further research will be directed at confirming and verifying the obtained results in terms of various types of distributions of the duration of individual activities and the impact of generating basic parameters of activity durations. Therefore, it is worth emphasizing that the obtained results depend on the adopted methodology, in particular the numerical procedure in point 3 (generating times: optimistic, modal, pessimistic) and point 5 (generating the duration of activities from a normal distribution with parameters calculated on the basis of point 3).

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