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APPLICATION OF THE TREFFTZ METHOD FOR OPTION PRICING

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Purpose: Option pricing is hardly a new topic, however, in many cases they lack an analytical solution. The article proposes a new approach to option pricing based on the semi-analytical Trefftz method.
Design/methodology/approach: An appropriate transformation makes it possible to reduce the Black-Scholes equation to the heat equation. This admits the Trefftz method (which has shown its effectiveness in heat conduction problems) to be employed. The advantage of such an approach lies in its computational simplicity and in the fact that it delivers a solution satisfying the governing equation.
Findings: The theoretical option pricing problem being considered in the paper has been solved by means of the Trefftz method, and the results achieved appear to comply with those taken from the Black-Scholes formula. Numerical simulations have been carried out and compared, which has confirmed the accuracy of the proposed approach.
Originality/value: Although a number of solutions to the Black-Scholes model have appeared, the paper presents a thoroughly novel idea of implementation of the Trefftz method for solving this model. So far, the method has been applied to problems having nothing in common with finance. Therefore the present approach might be a starting point for software development, competitive to the existing methods of pricing options.

Keywords: contract options, option pricing, Black-Scholes model, Trefftz method.

Category of the paper: Research paper.

1. Introduction

Creating innovative financial instruments which provide the investor with the possibility of risk reduction and hedging against unfavorable price movements in the underlying asset as well as securing above-average returns on investment are crucial for financial market development.

Option contracts play a special role among these instruments. All option contracts give holders the right, but not the obligation, to buy or sell the underlying instrument at a predetermined price and at a specified time.

Although it has been nearly 50 years since the seminal work on option pricing by Fischer Black, Myron Scholes and Robert Merton was published, dedicated research is underway to formulate a model that would allow fair option pricing, which is particularly relevant to financial research areas. Since the introduction of the Black-Scholes formula for option pricing, numerous analytical, numerical (using Monte Carlo simulations) methods and such that combine the previously mentioned approaches (i.e. analytical approximation models) have been developed.

Approximate methods of determining an option price were used in, among others, the works by M.J. Brennan and E.S. Schwartz (1977) and G. Courtadon (1982). Their idea was to solve the Black-Scholes differential equation numerically (they undertook one of the most challenging problems in derivatives, which is the valuation and optimal exercise of American options). J. Hull and A. White (1990) also adopted this approach. In the literature on the subject, simulation techniques (Boyle, 1977), estimation techniques (Duffie, and Glynn, 1996) and the martingale pricing approach were used.

The aim of the paper is to apply the Trefftz method to find an approximate solution to a differential equation describing the price of derivatives. The method has already been successfully applied to solving either direct or inverse problems in mechanics.

In general, a complete mathematical description of a physical phenomenon leads to an equation (usually a differential equation) which governs the process. Not only the equation needs to be established but also

- geometric properties (size and shape) of a body to which the equation applies,
- physical properties of the material,
- boundary and/or initial conditions.

If any of the above model elements is unknown, we deal with an inverse problem where, figuratively speaking, the effects are known while the causes have to be found. According to the classification proposed in (Ozisik, and Orlande, 2000), inverse problems include those concerning identification of

- shape of a body where the process takes place,
- boundary/initial values,
- source functions,
- material properties (parameter estimation).

The inverse problems like those addressed in (Beck, and Woodbury, 2016; Grysa, 2010; Maciąg, 2009; Ozisik, and Orlande, 2000) belong to ill-posed problems (Hadamard, 1902) and require efficient and robust solution methods. Such requirements are met by the numerical-analytical Trefftz method which can be applied even when the boundary or/and initial conditions are not fully known (Grysa, 2010; Maciąg, 2009).

In option pricing problems the issue is to determine the premium of the option which means that the initial condition has to be estimated. Hence we deal with an inverse problem. The present paper shows, to the best knowledge of the authors, the very first application of the Trefftz method to calculate an option exercise price. The solutions to the Black-Scholes equation for European put and call options obtained by the Trefftz method are compared with those taken from the Black-Scholes formula under the assumption of log-normal distribution in the pricing model.

The paper begins with a brief introduction to derivatives market, in particular European-style options. It provides the reader with essentials of the Black-Scholes model commonly used for option pricing. Also, it introduces the so-called Trefftz functions which we employ for the solution and it gives the general characteristics of the Trefftz method itself. The main part of the paper shows, based on numerical simulations, evaluation of European option prices using the Black-Scholes in comparison with the results by the Trefftz method. The obtained results are supposed to be the basis for further study aimed to develop original software for option pricing concerning more complicated cases than standard European options.

2. The idea of contract options

An option is a contract that gives the buyer the right to buy (long call) or sell (long put) an option and obliges the seller (short position) to deliver or receive a fixed amount of the underlying asset at a specified (fixed) price (the so-called strike price or exercise price) within a specified date (the so-called maturity or expiry date). The strike price is the basis for determining the option settlement amount – the relationship between an option strike price and the underlying stock's spot price determines the option value. Options are generally used for hedging purposes but can also be used for speculation.

One of the most important features that distinguishes option contracts from forward or futures contracts is the asymmetry of the payout profile (the value of an option at its expiry), i.e. the option holder has the right and not the obligation to exercise the option. In return for the acquired right (privilege), the option buyer pays the seller a so-called option premium. The value of the premium depends on many factors, in particular, on the fluctuations in the quotation of the underlying instrument being the subject of the transaction (the higher the volatility of the underlying instrument, the higher the option price) and the option strike price (the price at which the option is settled, the higher the call option exercise price, the lower its market price). In the case of a put option, the relationship is reversed – the higher the strike price, the more the option is worth.

Volatility is a measure of uncertainty or risk related to the size of changes in the underlying stock price (traditionally, it is viewed as synonymous with variance risk and can be thought of as a proxy of market risk). Volatility is also the most difficult factor to estimate. It determines the future volatility of the underlying asset during the life of the option. Therefore, forecasting volatility is the first step in valuating options. The literature on the subject describes many techniques developed to do this: a technique based on historical volatility (also known as realized volatility or statistical volatility), in which the volatility is determined on the basis of the historical prices of the underlying asset, and an implied volatility approach based on the current prices of transactions in options (Piontek, 2020; Rubaszek, 2012). Implied volatility represents the expectations of market participants towards a change in the underlying instrument. The algorithm for determining the implied volatility of the WIG20 index is available on the WSE website.

Another factor which impacts option valuation is interest rate (for a standard option pricing model like Black-Scholes, the risk-free one-year Treasury bills rates are used). Change in the risk-free interest rate impacts call and put option premiums inversely: calls benefit from rising rates while puts lose value. The opposite is true when interest rates fall.

It is worth noting that the risk for the buyer is limited to the amount of the premium and the potential profit may be unlimited, while the risk for the seller is unlimited and their potential profit is equal to the amount of the premium received.

Option contracts (derivative instruments in general) can trade over-the-counter (OTC) or on an exchange. The most common underlying assets for derivatives are stocks, currencies, bonds, commodities and interest rates.

The Warsaw Stock Exchange trades European-style options based on the WIG20 index (they debuted on the Warsaw bourse in September 2003). A characteristic feature of the European option is that its execution is restricted until the expiration date. However, it doesn't mean, that you cannot sell or buy options at any time, because there is a possibility of buying or selling to "close" the position prior to expiration (www.gpw.pl/pub/GPW/files/PDF/standardy_pl/Standard_opcje_WIG20, 2020).

3. Black–Scholes model

Determining the price of an option is used to estimate the fair value cost of an option under a given set of conditions at any moment of time. Due to the fact that the factors determining the price of an option, i.e. the value of the underlying instrument, its volatility, and the interest rate change randomly, the only way of knowing these parameters in advance is to use theoretical models (which are a simplified picture of reality). According to the literature, the search for rules governing the derivatives market began at the end of the 19th century, with the work of

L. Bachelier, R. Sage, C.M. Sprenkle, A.J. Boness (Weron, and Weron, 2008; Napiórkowski, 2002) and undoubtedly F. Black, M. Scholes and R. Merton.

The Black-Scholes model assumes that the price of traded assets follows a geometric Brownian motion with constant drift μ and volatility σ, hence the change in the total value of stock (pricing process) $\{S_t\}_t$ is given by:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

where:
S_t – current stock (or other underlying) price,
$\{W_t\}_t$ – Wiener process (or standard Brownian motion, contains the information about the randomness of the asset price),
t – time.

The Wiener process $\{W_t\}_t$ is a stochastic process defined by the following basic postulates:

- $W_0 = 0$,
- W_t is a stochastic process with independent normally distributed increments,

$\forall_{k\leq t}\, W_t - W_k \sim N(0, t-k)$, where $N(m, s^2)$ denotes the normal distribution with the expected value m and the variance s^2,

- the process paths are continuous, however, the path is fractal, and not differentiable anywhere (Jakubowski, and Sztencel, 2001).

Black and Scholes, when deriving their model, employ the following assumptions:

- there is no arbitrage (there is no opportunity of making a riskless profit);
- there exists a self-financing strategy;
- the interest rates are assumed to be constant during the option expiry period;
- stock prices are lognormally distributed;
- the stock price is continuous – can be modelled by Ito's continuous stochastic process;
- transactions costs and taxes are zero;
- no dividends are paid out during the life of the option;
- there is a possibility of a short sale.

The above assumptions made it possible to reduce the problem of option pricing to the solution of the following equation, known in the literature as the Black-Scholes partial differential equation

$$\frac{\partial Y(S_t,t)}{\partial t} = rY(S_t,t) - rS_t\frac{\partial Y(S_t,t)}{\partial S_t} - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 Y(S_t,t)}{\partial S_t^2} \tag{2}$$

with the boundary condition:

- for a call option

$$C_T = Y(S_T, T) = \begin{cases} S_T - K & \text{for } S_T - K \geq 0 \\ 0 & \text{for } S_T - K < 0 \end{cases} \tag{3}$$

- for a put option

$$P_T = Y(S_T, T) = \begin{cases} K - S_T & \text{for } K - S_T \geq 0 \\ 0 & \text{for } K - S_T < 0 \end{cases} \tag{4}$$

where:
r – risk-free interest rate,
T – maturity time,
$Y(S_t, t)$ – option price at time t,
K –strike price.

The above assumption about the lack of arbitrage and the existence of a self-financing strategy guarantees that it is possible to construct (from stocks and options) a risk-free portfolio (the so-called arbitrage portfolio) whose profitability is equal to the risk-free rate.

Under assumptions and constraints stated above the Black-Scholes equation has analytical solution for European options given by formulas

- the price of a call option

$$C_T = S_T \Phi(d_+) - e^{-rT} K \Phi(d_-) \tag{5}$$

- the price of a put option

$$P_T = -S_T \Phi(-d_+) + e^{-rT} K \Phi(-d_-) \tag{6}$$

where:
Φ is the cumulative normal distribution function;
d is determined from the formula

$$d_{\pm} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{7}$$

It is noteworthy that the Black-Scholes formula is correct when the short-term interest rate r is constant. In practice, it is assumed that the interest rate is equal to the risk-free interest rate for an investment with a maturity $T - t$ (this type of analysis is usually performed on the basis of a yield curve using the Nelson-Siegel model) (Rubaszek, 2012).

As mentioned above, at the time of pricing, all of the parameters used in the Black-Scholes model are clear and known; the only one that is not known with certainty (is not deterministic) is volatility σ. In practice, to estimate σ, it is usually assumed that future volatility equals historical volatility. A more advanced analysis of volatility consists in determining it based on building time series models (especially the ARCH class models), deriving the distribution of the volatility, or using the fundamental analysis of the factors determining the price of the basic instrument and making an expert forecast (Rubaszek, 2012). Recently a Black-Scholes model with GARCH volatility has been introduced (Gong et al., 2010; Kamiński, 2013). In the presented approach, we assume that the value of this parameter is known.

Upon using the following substitutions in equation (2)

$$u = \frac{2}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)\left[ln\left(\frac{S_t}{K}\right) - \left(r - \frac{1}{2}\sigma^2\right)(t-T)\right] \tag{8}$$

$$\tau = -\frac{2}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)(t-T) \tag{9}$$

and with the function $Y(S_t, t)$ expressed in the form

$$Y(S_t, t) = e^{r(t-T)} y(u, \tau) \tag{10}$$

finding the solution of (2) comes down to solving the following partial differential equation

$$\frac{\partial y(u,\tau)}{\partial \tau} = \frac{\partial^2 y(u,\tau)}{\partial u^2} \tag{11}$$

and whose initial conditions are dependent on the type of options:

- for a put option

$$y(u,0) = \begin{cases} K\left[exp\left(\frac{\frac{1}{2}u\sigma^2}{r-\frac{1}{2}\sigma^2}\right) - 1\right] & \text{for } u \geq 0 \\ 0 & \text{for } u < 0 \end{cases} \tag{12}$$

- for a call option

$$y(u,0) = \begin{cases} K\left[1 - exp\left(\frac{\frac{1}{2}u\sigma^2}{r-\frac{1}{2}\sigma^2}\right)\right] & \text{for } u \leq 0 \\ 0 & \text{for } u > 0 \end{cases} \tag{13}$$

It is worth noticing that equation (11) is of parabolic type and is usually interpreted as the heat conduction equation.

4. Trefftz method

Equation (11) describing the price of an option has been solved using various techniques (Sawangtong et al., 2018), e.g. the finite difference method or the finite element method, which represent purely numerical methods but also the Adomian decomposition method, homotopy perturbation method, radial basis function method, etc.

The present approach employs the numerical-analytical Trefftz method. The logic of this method is to approximate the solution of a differential equation with a linear combination of certain basis functions (named T-complete functions or Trefftz functions) satisfying the given differential equation. The origins of the method date back to the 1920s when a German mathematician E. Trefftz (1926) used it for solving the Laplace equation. Since then, a few versions of this method have appeared and been applied to various types of partial differential equations but the main concept remains unchanged. It might be worth to note that the method is normally applicable only to homogeneous linear differential equations since only for such equations we can generate T-complete functions. In order to employ the Trefftz method for solving nonlinear differential equations or nonhomogeneous linear equations, we have to combine it with some other methods, e.g. with the finite element method (Maciejewska, 2017), the Picard method (Grabowski et al., 2018) or the homotopy method (Hożejowska, 2015). However, we have to accept that the solution will not satisfy the governing equation exactly so it will be deprived of its advantage.

For equation (11) describing the considered option pricing problem, the appropriate T-complete functions are the so-called heat polynomials. Introduced and investigated by P.C. Rosenbloom and D.V. Widder (1959), they are expressed with a formula

$$v_n(u,\tau) = n! \sum_{k=0}^{n} \frac{u^{n-2k}\tau^k}{(n-2k)!k!} \tag{14}$$

where n is used to denote the n-th polynomial.

According to logic of the Trefftz method, the solution of equation (11) has to be approximated with a linear combination of the heat polynomials which gives

$$y(u,\tau) = \sum_{n=0}^{N} a_n v_n(u,\tau) \tag{15}$$

where N denotes the number of the T-complete functions employed, excluding a constant function v_0.

Unknown coefficients a_n of the linear combination (15) will be specified to ensure best fulfilment of the imposed boundary conditions (12) or (13) and more precisely, to minimise the squared differences between the exact and computed values of the function $y(u,\tau)$. Hence an approximate solution of the Black-Scholes equation (2) obtained by the Trefftz method can be expressed by the following formula

$$Y^*(S_t,t) = e^{r(t-T)} \sum_{n=0}^{N} a_n v_n(u,\tau) \tag{16}$$

in which an approximate value of a European call (or put) option C_t^* (or P_t^*) comes from the solution of equation (11) with respective initial conditions 12 (or 13). Since we use the Trefftz method in both cases, then, consequently, the solutions C_t^* and P_t^* will satisfy the Black-Scholes equation exactly but the boundary conditions will be satisfied only in sense of least-squares.

The Trefftz method enjoys little popularity among economists. Therefore, it seems reasonable to shortly outline its most characteristic features. In terms of merits, it:

- delivers continuous solutions which fulfill the appropriate governing equation,
- is robust to small uncertainties (or disturbances) of the input data,
- can be applied both to direct and inverse problems,

– allows various types of imposed conditions, including overspecified or partly missing,
– does not require time-consuming mesh generation (as e.g. in FEM),
– does not lead to complex computations (only integration or differentiation of simple functions performed only along the domain boundary) and therefore does not require advanced software and hardware products, unless for problems with highly complicated geometry.

In terms of disadvantages, the Trefftz method:

– can be directly applicable only to homogeneous linear differential equations,
– is prone to Runge's phenomenon (oscillation of solution at the edges) which might occur at high order of approximations.

5. Results of computation

The numerical simulations below consider call and put options and are based on the data taken from (Cervera, 2019). The calculations assumed the strike price $K = 10$, expiration date in 3 months (i.e. $T = 0.25$), 3-month risk-free rate $r = 0.1$, volatility $\sigma = 0.4$ and the various possible values of the underlying asset price S_t from $0.1K = 1$ to $2K = 20$, (Cervera, 2019). The calculations were performed for 10 T-complete functions. Figure 1 compares call option price C_t^* obtained by the Trefftz method (Figure 1a) with the corresponding price C_t (Figure 1b) provided by the Black-Scholes formula, both presented versus the price of the underlying asset S_t.

Figure 2 presents a 3D graph of the call option price C_t^*, obtained by the Trefftz method as a function of the price of the underlying S_t and time t.

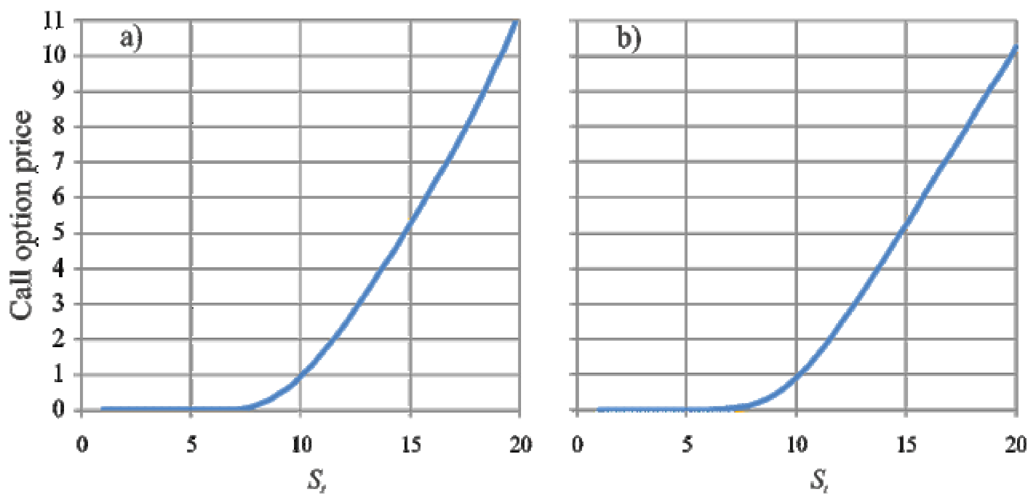

Figure 1. The call option price obtained by: a) the Trefftz method and b) the Black-Scholes formula as a function of the changing price of the underlying instrument and constant volatility σ = 0.4.

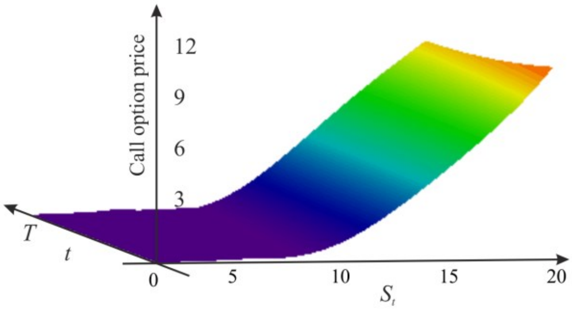

Figure 2. The value of the call option price C_t^* depending on the price of the underlying S_t and time t.

Similar calculations were performed for the put option price, Figure 3.

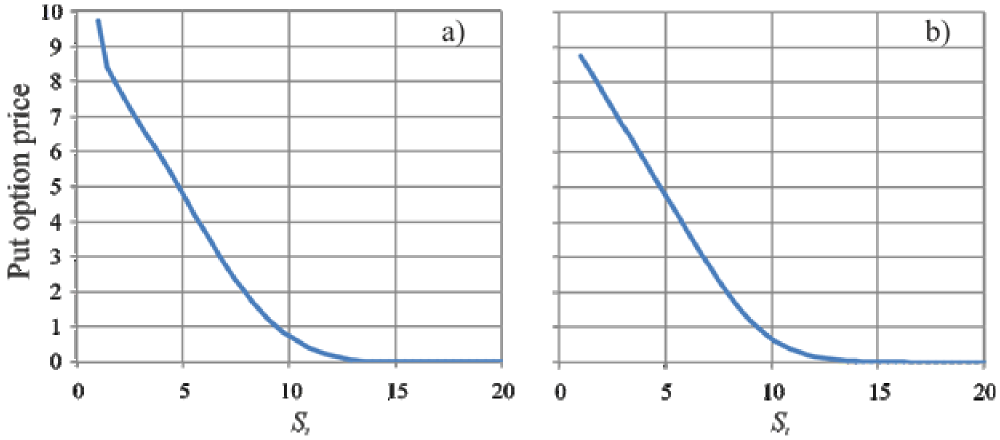

Figure 3. The put option price obtained by: a) the Trefftz method and b) the Black-Scholes formula as a function of the changing price of the underlying instrument and constant volatility σ = 0.4.

The impact of volatility σ on option prices has also been examined. The performed calculations for the following parameter values σ = 0.4; 0.3; 0.2; 0.1; 0.05 showed no significant differences for both types of options, regardless of the calculation method.

Figure 4 shows the differences $C_t^* - C_t$ (Figure 4a) and $P_t^* - P_t$ (Figure 4b) concerning the call and put options respectively. The differences for some selected values of volatility σ are plotted as functions of the underlying asset price S_t.

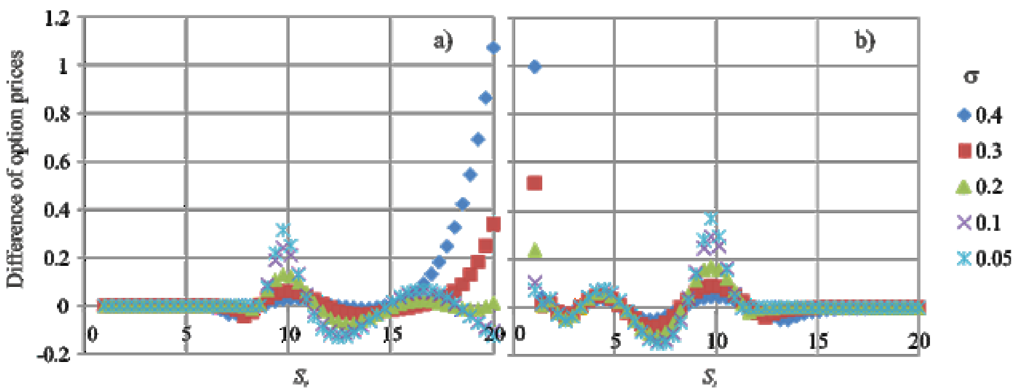

Figure 4. Price differences: a) for call option $C_t^* - C_t$, b) for put option $P_t^* - P_t$.

As follows from Fig. 4, the Trefftz method gave the results similar to those obtained from the Black-Scholes formula with a maximum difference of 0.99 for the put option and 1.07 for the call option.

The obtained results indicate the following regularities found in the presented methods of option valuation (Trefftz vs. Black-Scholes).

- The largest discrepancies between the results appear: at the ends of the price range of the underlying instrument (for call options) and at the beginning of the range (for put options), which might require complementing the model with some additional boundary conditions.
- The observed discrepancies attain a local extremum in case the current underlying asset price S_t equals the strike option price K.

6. Conclusions

Option pricing can be identified as an inverse problem where the unknown initial condition, namely – the option price at time 0 – has to be somehow determined. The paper presents the first approach to option pricing with the use of the Trefftz method. As demonstrated by the provided examples, the Trefftz method allows to obtain an approximate solution to the option pricing problem with satisfactory accuracy. Generally, it provides astonishingly similar results to those achieved by the Black-Scholes formula. More specifically, the differences have been very close to zero ($\sim 10^{-2}$ or less) on more than 82% of the analyzed price range of the underlying instrument S_t. The largest discrepancies between the results occurred for the highest volatility σ at the ends of the underlying instrument's price range (for call options) and at the beginning of the range (for put options).

The authors would like to emphasize that the proposed approach provides promising results. Due to its special features, the Trefftz method might be competitive against the methods used so far in solving more complicated mathematical models referring to some other types of options.

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